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Tachyon backgrounds in 2D string theory

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Abstract: We consider the construction of tachyonic backgrounds in two-dimensional string theory, focusing on the Sine-Liouville background. This can be studied in two different ways, one within the context of collective field theory and the other via the formalism of Toda integrable systems. The two approaches are seemingly different. The latter involves a deformation of the original inverted oscillator potential while the former does not. We perform a comparison by explicitly constructing the Fermi surface in each case, and demonstrate that the two apparently different approaches are in fact equivalent.

Keywords: Conformal Field Models in String Theory, Bosonic Strings, String Duality, Matrix Models.

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Contents

1. Introduction

Recent developments in string theory have taught us the importance of holography in trying to understand gravity and its relationship to conformal field theories (CFTs). A prime example of the duality between gravitational physics and a lower-dimensional theory not containing gravity is given by the AdS/CFT correspondence [1]. However, the earliest example of a theory admitting such a holographic description is two-dimensional (2D) non-critical string theory (see [2–6] for reviews). In this case the duality is between one-dimensional $c = 1$ matrix quantum mechanics and two-dimensional string theory, realized as a collective field theory. This has also been understood as an example of open-closed string duality [7–9].

Although 2D string theory is just a toy model, it has proven to be very useful because of its solvability. It is, in fact, an example of a sigma-model with integrable dynamics. It also allows us to address other relevant string theory issues, such as time dependence and the search for new backgrounds, within the context of a model that is more tractable.

In 2D string theory the only dynamical degree of freedom is the massless scalar tachyon, whose dynamics (scattering amplitudes) has been shown to be well described by the collective field theory of a single scalar [10]. Furthermore, the effective (collective) field theory of the tachyon is exactly solvable [11].

Most of the 2D string theory investigations have involved the so-called Liouville background, described in the CFT by a $c = 1$ matter field coupled to the Liouville field. Additional backgrounds can be obtained by replacing the Liouville term by a general tachyon (vertex) perturbation. Such perturbed backgrounds have been studied within the context of collective field theory [12] as well as by Alexandrov, Kazakov and Kostov (AKK) via the formalism of Toda integrable systems [13, 14]. Also, a lot of effort has been recently devoted to the study of cosmological issues in certain time-dependent backgrounds [15 – 22].

In this work we will concentrate on the Sine-Liouville background, as an example of the tachyon perturbations mentioned above, and we will compare the two seemingly different ways in which it has been addressed. In the collective field approach of [12], the hamiltonian was that of the standard inverted harmonic oscillator, while in the work of AKK the Sine-Lioville potential was simulated by introducing certain *deformed* hamiltonians. At the moment it is not clear what the correspondence between these two different approaches is. It is the purpose of this paper to clarify this issue. In particular, we will explicitly construct the Fermi surface for each distinct approach, and show that the two methods are equivalent. We should mention that some of the simpler backgrounds which were recently studied in [15 – 22] were explicitly shown to arise as solutions of the standard inverted oscillator collective field theory.

The paper is organized as follows. In section 2 we describe certain 2D string theory backgrounds, and we introduce the Sine-Liouville model as a perturbation of Liouville theory. Section 3 contains the basics of collective field theory, as well as the collective approach to building non-trivial backgrounds. This construction is then applied to the case of the Sine-Liouville background, and it is used to obtain the explicit shape of the corresponding Fermi surface. Section 4 outlines the method of [13, 14] for constructing new backgrounds. This serves as a starting point for extracting the Sine-Liouville Fermi surface, but in a manner entirely different from that of the collective field theory. Section 5 consists of a comparison of the two approaches, and shows agreement of the Fermi surfaces.

2. 2D string theory backgrounds

One way to discuss the dynamics of strings is through the β-function approach, which provides the effective field theory description of the low-energy fields. We would like to start by reviewing the connection between the nonlinear σ-model and the corresponding effective theory.

The 2D nonlinear σ-model is given by

$$S_\sigma = \frac{1}{4\pi}\int d^2\sigma\sqrt{g}\,[g^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) + R^{(2)}D(X) + T(X) + \cdots]\,, \tag{2.1}$$

where $X \equiv X^\mu = (X^0, \varphi)$ parametrize the two-dimensional target space, $T(X)$ is the massless tachyon, $D(X)$ is the dilaton and $G_{\mu\nu}(X)$ the graviton. The vanishing of the β-functions guarantees conformal invariance, and yields the spacetime equations of motion. As is well known, the β-function equations can be shown to follow from the action

$$S_{\rm eff} = \frac{1}{2\pi\kappa^2}\int d^2X\sqrt{G}e^{-2D(X)}\left[R + 4(\nabla D)^2 - (\nabla T)^2 + 2T^2 - V(T)\right], \tag{2.2}$$

where κ is the string coupling and $V(T)$ is the tachyon potential, which includes tachyon interactions and which we leave unspecified.

One solution ensuring the vanishing of the β-functions is the (euclidean) *linear dilaton vacuum*, given by

$$\begin{aligned} G_{\mu\nu} &= \delta_{\mu\nu}\,, \\ D(X) &= \sqrt{2}\varphi\,, \\ T(X) &= 0\,. \end{aligned} \tag{2.3}$$

In order to find more general solutions, let us consider the *linearized* equation of motion for the tachyon,

$$\beta^T = -2\nabla^2 T + 4\nabla D\nabla T - 4T = 0\,. \tag{2.4}$$

With the above choice for the dilaton, $D(X) = \sqrt{2}\varphi$, eq. (2.4) becomes

$$(\partial^2_{X^0} + \partial^2_\varphi - 2\sqrt{2}\partial_\varphi + 2)T = 0\,. \tag{2.5}$$

The linearized *static* tachyon equation $(\partial^2_\varphi - 2\sqrt{2}\partial_\varphi + 2)T = 0$ has two linearly independent solutions, $e^{\sqrt{2}\varphi}$ and $\varphi e^{\sqrt{2}\varphi}$. One can show that by taking

$$\begin{aligned} G_{\mu\nu} &= \delta_{\mu\nu}\,, \\ D(X) &= \sqrt{2}\varphi\,, \\ T(X) &= \frac{\mu}{4}e^{\sqrt{2}\varphi} \end{aligned} \tag{2.6}$$

we obtain another consistent string theory solution. This choice specifies the so-called Liouville background,

$$S_0 = \frac{1}{8\pi}\int d^2\sigma\left[(\partial X^0)^2 + (\partial\varphi)^2 + 2\sqrt{2}\varphi R^{(2)} + \mu e^{\sqrt{2}\varphi}\right], \tag{2.7}$$

a $c = 1$ conformal field theory coupled to the Liouville field φ. The central charge $c \equiv c_X = 1$ refers to the matter coordinate X^0, while the Liouville field with $Q = 2\sqrt{2}$ carries a central charge $c_\varphi = 1 + 3Q^2 = 25$, giving us the expected total charge $c_{X^0} + c_\varphi = 26$. So far we have only considered static solutions of the tachyon linearized equation. If we allow for time dependence, we can see that tachyons of the form $T = e^{(\sqrt{2}\pm p)\varphi}\, e^{\pm ipX^0}$ are also solutions of (2.4). In particular, the specific linear combination $T = e^{(\sqrt{2}-|p|)\varphi}\cos(pX^0)$ is what is usually referred to as the Sine-Liouville background.

Liouville theory can also be obtained in the following way. Start from the Euclidean string action

$$S_0 \sim \int d^2\sigma\sqrt{detg}\left(g^{ab}\partial_a X^0\partial_b X^0 + \mu\right), \tag{2.8}$$

with g^{ab} the world-sheet metric, μ the "cosmological constant" and $X(\sigma)$ the embedding of the string into Euclidean time. Choosing the conformal gauge $g^{ab} = e^\varphi \hat{g}^{ab}$, the dilaton φ becomes dynamical (due to the conformal anomaly), and the world-sheet CFT action takes the Liouville form

$$S_0 \sim \int d^2\sigma\left[(\partial X^0)^2 + (\partial\varphi)^2 + QR^{(2)}\varphi + \mu e^{\gamma\varphi}\right]. \tag{2.9}$$

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The background charge Q is determined by the usual central charge requirement $c_\varphi = 1 + 3Q^2 = 25$, while the Liouville exponent γ is found by imposing that the cosmological term perturbation is marginal, $-\frac{1}{2}\gamma(\gamma - Q) = 1$. From this perspective, the Liouville interaction term is thought of as the insertion of a vertex operator, $\delta L = \mu V_\gamma$, with $V_\gamma = e^{\gamma\varphi}$.

More general perturbations can be considered. In fact, if the Euclidean time is compactified with radius R, the spectrum of admissible momenta becomes discrete, $p_n = n/R$, $n \in \mathbb{Z}$. In addition, there is also a discrete spectrum of operators describing winding modes on the world-sheet, with charges $q_m = mR$, $m \in \mathbb{Z}$. The form of *vertex* operators V_p describing momentum modes, and *vortex* operators $\tilde{V}_q$ describing winding modes is

$$V_p \sim \int d^2\sigma e^{ipX_0} e^{(\sqrt{2}-|p|)\varphi}\,, \tag{2.10}$$

$$\tilde{V}_q \sim \int d^2\sigma e^{iq\tilde{X}_0} e^{(\sqrt{2}-|q|)\varphi}\,, \tag{2.11}$$

where $\tilde{X}^0$ is the dual coordinate to X^0. One can obtain new backgrounds of the compactified 2D string theory by perturbing the Liouville action with both vertex and vortex operators:

$$S = S_0 + \sum_{n\neq 0}(t_n V_n + \tilde{t}_n \tilde{V}_n)\,. \tag{2.12}$$

In particular, considering only momentum mode perturbations and choosing $n = \pm 1$, we reproduce the Sine-Liouville background (with $p_1 = \frac{1}{R}$) :

$$S \sim \int d^2\sigma \left[(\partial X^0)^2 + (\partial\varphi)^2 + QR^{(2)}\varphi + \mu e^{\gamma\varphi} + \lambda e^{(\sqrt{2}-\frac{1}{R})\varphi}\cos\left(\frac{X^0}{R}\right)\right], \tag{2.13}$$

where the couplings have been taken to be $t_1 = t_{-1} = \frac{\lambda}{2}$.

3. The collective field theory approach

A correspondence between one-dimensional matrix quantum mechanics and two-dimensional string theory can be established by using the formalism of collective field theory (see [4] for a review). Consider the theory of a hermitean $N \times N$ matrix $M(t)$ in one time dimension, with lagrangian

$$L = \frac{1}{2}\,\mathrm{Tr}\left(\dot{M}^2 - u(M)\right). \tag{3.1}$$

The non-critical $c = 1$ string theory corresponds to choosing an inverted harmonic oscillator potential, $u(M) = -\frac{M^2}{2}$. There is an associated $\mathrm{U}(N)$ conserved charge $J = i[M, \dot{M}]$. In the singlet sector $J|\,\rangle = 0$ the model becomes a gauge theory, and the matrix M can be diagonalized, $M(t) \to diag(\lambda_i(t))$. It can be shown that the $\mathrm{U}(N)$ invariant sector of the matrix model is described by a non-relativistic quantum mechanics of free fermions - the eigenvalues of the matrix. Collective variables for the model are introduced in the following way:

$$\phi_k(t) = \mathrm{Tr}\left(e^{ikM(t)}\right) = \sum_{i=1}^{N} e^{ik\lambda_i(t)}\,. \tag{3.2}$$

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Notice that so far we have been dealing with a one-dimensional (matrix) theory, depending only on time. A two-dimensional model arises naturally when one introduces an additional spatial dimension x, which is related to the space of eigenvalues λ_i of the matrix $M(t)$:

$$\phi(x,t) = \frac{1}{2\pi}\int dk\, e^{-ikx}\,\phi_k(t) = \sum_{i=1}^{N}\delta(x-\lambda_i(t))\,. \tag{3.3}$$

Thus, the collective field has the physical interpretation of a density field of fermions. Finally, introduction of a conjugate field Π_ϕ with Poisson brackets $\{\phi(x),\Pi_\phi(y)\} = \delta(x-y)$ yields a canonical phase space.

One important difference between the matrix model and the string theory side which will be evident soon is that, while the collective equations of motion are highly nonlinear, the matrix equations (for the oscillator potential) are linear. This feature of integrability, or exact solvability, is one of the main advantages of applying the matrix model formalism to the study of 2D string theories. It is precisely through the nonlinear transformation $\phi(x,t) = \mathrm{Tr}\,\delta(x-M(t))$ that the matrix model provides an exact solution to the corresponding nonlinear field theory.

The dynamics of the collective field theory can be induced from that of the matrix model variables $M(t)$ and $P(t) = \dot{M}(t)$. At the classical level the collective action is given by

$$S = \int dx\, dt\left[\frac{1}{2}\frac{(\partial_x^{-1}\dot{\phi})^2}{\phi} - \frac{\pi^2}{6}\phi^3 + \frac{1}{2}(x^2+\mu)\phi\right]. \tag{3.4}$$

The cosmological constant μ is introduced as a "chemical potential" coupled to the size of the matrix. It acts as a Lagrange multiplier enforcing the normalization condition $\int dx\,\phi(x,t) = N$. We can eliminate the explicit μ dependence, which we do by properly rescaling x and ϕ, $x \to g^{-1/2}x$, $\phi \to g^{-1/2}\phi$, and setting $g\mu = 1$.

The collective field $\phi(x,t)$ and its conjugate momentum Π_ϕ can be written in terms of left- and right-moving fields $p_\pm(x,t)$,

$$p_\pm(x,t) = g^2\partial_x\Pi_\phi(x,t) \pm \pi\phi(x,t), \tag{3.5}$$

in terms of which the action becomes

$$S = -\frac{1}{2\pi g^2}\int dxdt\left[\frac{1}{2}\left(p_+\partial_x^{-1}\dot{p}_+ - p_-\partial_x^{-1}\dot{p}_-\right) + \frac{1}{6}\left(p_+^3 - p_-^3\right) - \frac{1}{2}(x^2+1)\left(p_+ - p_-\right)\right]. \tag{3.6}$$

The equation of motion for the fields $p_\pm$ is

$$\partial_t p_\pm = x - p_\pm\partial_x p_\pm \tag{3.7}$$

whose static solution is given by $p_\pm = \pm\sqrt{x^2+1}$. We expand the fields around the static background,

$$p_\pm = \pm\sqrt{x^2+1} + \eta_\pm\,, \tag{3.8}$$

where the fluctuations $\eta_\pm$ are in general time-dependent. It will be convenient to introduce a new coordinate τ such that $x = \sinh(\tau)$, and define $\alpha_\pm = \eta_\pm\cosh(\tau)$. The action

$S = S^+ + S^-$ then becomes

$$S^{\pm} = \mp\frac{1}{4\pi}\int dtd\tau\left[-\alpha_{\pm}\partial_{\tau}^{-1}\dot{\alpha}_{\pm} + \alpha_{\pm}^2 + \frac{g}{3}\frac{\alpha_{\pm}^3}{\cosh(\tau)^2}\right]. \tag{3.9}$$

Since $\alpha_{\pm}$ are decoupled, it is enough to study only one of them; we will consider $\alpha \equiv \alpha^+$. Its equation of motion is

$$(\partial_t + \partial_\tau)\alpha = -\frac{g}{2}\partial_\tau\left(\frac{\alpha^2}{\cosh(\tau)^2}\right), \tag{3.10}$$

which can be solved perturbatively in g. Let α_0 be the solution of the free equation, $(\partial_t + \partial_\tau)\alpha_0 = 0$; one can then solve for α in terms of the linearized field α_0,

$$\alpha = \alpha_0 + g\,\alpha_1 + g^2\,\alpha_2 + \cdots . \tag{3.11}$$

Note that α and α_0 are connected through the relation

$$\alpha_0(t,\tau) = \alpha(t,\tau) - \int dt'\,d\tau'\,\Delta_F(t-t',\tau-\tau')\,(\partial_{t'} + \partial_{\tau'})\,\alpha(t',\tau'), \tag{3.12}$$

where Δ_F is the Feynman propagator. It will also turn out to be useful to introduce in and out fields $\alpha_{\mathrm{in,\,out}}$

$$\alpha_{\mathrm{in,\,out}}(t,\tau) = \alpha(t,\tau) - \int dt'\,d\tau'\,\Delta_{R,\,A}(t-t',\tau-\tau')\,(\partial_{t'} + \partial_{\tau'})\,\alpha(t',\tau'), \tag{3.13}$$

with $\Delta_{R,\,A}$ retarded and advanced propagators, and to note that

$$\begin{aligned}\lim_{t\to-\infty}\alpha(t,\tau) &= \lim_{t\to-\infty}\alpha_{\mathrm{in}}(t,\tau)\\ \lim_{t\to\infty}\alpha(t,\tau) &= \lim_{t\to\infty}\alpha_{\mathrm{out}}(t,\tau).\end{aligned} \tag{3.14}$$

Finally, using properties of propagators, one can show that the Fourier transforms of these fields are related by

$$\tilde{\alpha}_0(w,k) = \theta(k)\tilde{\alpha}_{\mathrm{out}}(w,k) + \theta(-k)\tilde{\alpha}_{\mathrm{in}}(w,k). \tag{3.15}$$

The relations above are important because they uniquely determine α in terms of α_0; furthermore, any one of α_0, α_{in} or α_{out} contains all the information about the boundary values of the system.

It is precisely the linearized collective field α_0 that is identified with the world-sheet tachyon $T(X_0,\varphi)$. A quick way to see this is the following. Note that $(\partial_t + \partial_\tau)\alpha_0 = 0$ can be rewritten as $(-\partial_t^2 + \partial_\tau^2)\alpha_0 = 0$. Also, recall that the tachyon obeyed the linearized equation

$$(\partial_{X_0}^2 + \partial_\varphi^2 - 2\sqrt{2}\partial_\varphi + 2)T = 0. \tag{3.16}$$

After defining a new field S given by $T = e^{\sqrt{2}\varphi}S$, the above equation of motion becomes

$$(-\partial_t^2 + \partial_\varphi^2)S = 0, \qquad \text{after} \qquad X_0 = it, \tag{3.17}$$

showing that the rescaled tachyon S obeys the same equation as α_0. Thus, the relevant collective field theory quantity which we need to identify when trying to make a connection with a given tachyon background is going to be α_0.

An *exact* solution of the nonlinear classical equation (3.7) was obtained in parametric form in [11]. Using σ to parametrize x and p, we have

$$x = a(\sigma)\sinh(t-\sigma),$$
$$p = a(\sigma)\cosh(t-\sigma)\,,$$

obeying

$$p^2 - x^2 = a^2(\sigma) = 1 - g\,\epsilon(\sigma)\,, \tag{3.18}$$

where the new field $\epsilon(\sigma)$ denotes fluctuations about the static background. It is precisely this equation which specifies the profile of the Fermi sea, i.e. the Fermi surface. We can see from (3.18) that it is $\epsilon(\sigma)$ which encodes the essential information about a given background; thus, specifying $\epsilon(\sigma)$ corresponds to specifying the background completely. Since, as we have just seen, the world-sheet tachyon T can be related to α_0, in order to find the Fermi surface corresponding to a given background, we need to express $\epsilon(\sigma)$ in terms of α_0. This was done in [12]. We will briefly review the arguments of that work.

We start by considering a point on the Fermi surface with coordinate x. We evolve it back in time to $t \to -\infty$, and turn off the interaction, by setting $g = 0$. We then follow it back to present time, and label it by $x_{\rm in}$. Since $x = \sinh(\tau) = \sqrt{1-g\epsilon}\sinh(t-\sigma)$, we also have $x_{\rm in} = \sinh(\tau_{\rm in}) = \sinh(t-\tilde{\sigma})$. Combining these two expressions gives the equation

$$\tau(t=-\infty,\sigma) = \tau_{\rm in}(t=-\infty,\tilde{\sigma})\,, \tag{3.19}$$

which implies

$$\tilde{\sigma} = \sigma + \frac{\ln(1-g\epsilon)}{2}\,. \tag{3.20}$$

This gives

$$\tau_{\rm in}(t,\tilde{\sigma}) = t - \tilde{\sigma} = t - \sigma - \frac{1}{2}\ln(1-g\epsilon(\sigma))\,. \tag{3.21}$$

In the $t \to -\infty$ limit, it can be shown [12] that $\alpha = -\frac{\epsilon(\sigma)}{2} + \mathcal{O}(e^{2(t-\sigma)})$. This allows us to write, using (3.14),

$$\begin{aligned}\epsilon(\sigma) &= -2\lim_{t\to-\infty}\alpha(t,\tau) = -2\lim_{t\to-\infty}\alpha_{\rm in}(t-\tau_{\rm in}) \\ &= -2\alpha_{\rm in}\left(\sigma + \frac{1}{2}\ln(1-g\epsilon(\sigma))\right) = \epsilon_{\rm in}\left(\sigma + \frac{1}{2}\ln(1-g\epsilon(\sigma))\right),\end{aligned} \tag{3.22}$$

where we have used (3.21) and defined $\epsilon_{\rm in} \equiv -2\alpha_{\rm in}$. Similarly one can obtain an expression for $\epsilon(\sigma)$ in terms of $\epsilon_{\rm out} \equiv -2\alpha_{\rm out}$,

$$\epsilon(\sigma) = \epsilon_{\rm out}\left(\sigma - \frac{1}{2}\ln(1-g\epsilon(\sigma))\right). \tag{3.23}$$

These two expressions can be combined to directly relate $\epsilon_{\rm in}$ and $\epsilon_{\rm out}$:

$$\epsilon_{\rm in}(\sigma) = \epsilon_{\rm out}\left(\sigma - \ln(1-g\epsilon_{\rm in}(\sigma))\right). \tag{3.24}$$

The solution of the equation above is

$$\tilde{\epsilon}_{\text{in,out}}(k) = -\frac{1}{2\pi g}\frac{1}{1\mp ik/2}\int_{-\infty}^{\infty} d\sigma e^{-ik\sigma}\left[(1-g\epsilon(\sigma))^{1\mp ik/2}-1\right] \tag{3.25}$$

with $\tilde{\epsilon}_{\text{in, out}}(k)$ denoting Fourier transforms. This formula summarizes the strategy for constructing non-trivial backgrounds. Classical solutions can be characterized by their asymptotic behavior, which corresponds to specification of a linearized solution. In the above integral formula, this is equivalent to specifying $\tilde{\epsilon}_{\text{in, out}}(k)$. The exact solution can then be found by solving for $\epsilon(\sigma)$. Next, we will focus on the special case of the Sine-Liouville background.

3.1 The Sine-Liouville background

So far the analysis has been general. To apply it to the Sine-Liouville background, we need to specify $\tilde{\epsilon}_{\text{in,out}}(k)$. Define the Fourier transform $\tilde{T}(k)$ of the tachyon field in the following way,

$$T(X) = \int dk\, \tilde{T}(k)e^{ikX+\beta_k\varphi}\,, \tag{3.26}$$

where $\beta_p \equiv \sqrt{2}(1-\frac{|p|}{2})$. [1] In [12] it is shown that $\tilde{T}(k)$ should be identified with the Fourier transform of the collective field, $\tilde{\alpha}_0(k)$.

Plugging in the expression for the Sine-Liouville tachyon, $T(X_0,\varphi) = \lambda e^{\beta_p\varphi}\cos(pX)$, one finds

$$\tilde{\alpha}_0(k) = \frac{\lambda}{2}(\delta(k+p)+\delta(k-p))\,. \tag{3.27}$$

However, in order to allow for more general linear combinations of the linearized solutions, we take

$$\tilde{\epsilon}_0(k) \equiv \tilde{\alpha}_0(k) = \alpha\delta(k+p)+\beta\delta(k-p)\,. \tag{3.28}$$

Finally, using (3.15), one gets

$$\tilde{\epsilon}_{\text{in}}(k) = f(k)\theta(k)+\alpha\delta(k+p) \tag{3.29}$$

$$\tilde{\epsilon}_{\text{out}}(k) = f(k)\theta(-k)+\beta\delta(k-p)\,. \tag{3.30}$$

The unknown functions $f(k)$ and $\epsilon(\sigma)$ can be solved by plugging $\tilde{\epsilon}_{\text{in, out}}(k)$ into (3.25). This was done by [12] to several orders in perturbation theory, but here we will stop at cubic order:

$$\epsilon_0 = \alpha e^{-k\sigma}+\beta e^{k\sigma}, \tag{3.31}$$

$$\epsilon_1 = \frac{k}{2}(\alpha^2 e^{-2k\sigma}+\beta^2 e^{2k\sigma}), \tag{3.32}$$

$$\epsilon_2 = \left(\frac{3k^2}{8}+\frac{k}{4}\right)\left(\alpha^3 e^{-3k\sigma}+\beta^3 e^{3k\sigma}\right)+\left(\frac{k^2}{8}+\frac{k}{4}\right)\left(\alpha^2\beta e^{-k\sigma}+\beta^2\alpha e^{k\sigma}\right). \tag{3.33}$$

By making use of this result, we will now explicitly construct the Fermi surface.

[1] Notice that the fields X^0 and φ are related to those of section 2 by simple rescalings.

3.2 Constructing the Fermi surface

Plugging $\epsilon(\sigma)$ above into the generic Fermi surface (3.18) we find

$$\begin{aligned} a \sim 1 - \frac{g}{2}\left[\alpha e^{-k\sigma} + \beta e^{k\sigma}\right] - \frac{g^2}{8}\left[(1+2k)(\alpha^2 e^{-2k\sigma} + \beta^2 e^{2k\sigma}) + 2\alpha\beta\right] + \\ + g^3\left[\left(-\frac{c_2}{2} - \frac{k}{8} - \frac{3}{16}\right)(e^{-k\sigma}\alpha^2\beta + e^{k\sigma}\alpha\beta^2) + \right. \\ \left. + \left(-\frac{c_1}{2} - \frac{k}{8} - \frac{1}{16}\right)(\alpha^3 e^{-3k\sigma} + \beta^3 e^{3k\sigma})\right] + \mathcal{O}(g^4)\,, \end{aligned} \tag{3.34}$$

with $c_1 = \frac{3k^2}{8} + \frac{k}{4}$, $c_2 = \frac{k^2}{8} + \frac{k}{4}$. In order to express (3.18) entirely in terms of p and x, we need to find $\sigma = \sigma(p, x)$. It turns out to be more convenient to work in light-cone coordinates, which are typically taken to be

$$\begin{aligned} x_+ &= p + x = a(\sigma)e^{t-\sigma}\,, \\ x_- &= p - x = a(\sigma)e^{-t+\sigma}\,. \end{aligned}$$

However, in view of the comparison with the work of [13, 14], we choose to incorporate the explicit time dependence into the definition of new light-cone coordinates $X_\pm$, given by

$$X_+ = e^{-t}x_+ = a(\sigma)e^{-\sigma}\,, \tag{3.35}$$

$$X_- = e^{t}x_- = a(\sigma)e^{\sigma}\,. \tag{3.36}$$

Notice from (3.35), (3.36) that one can solve for σ as a function of either X_+ or X_-; this is a simple restatement of the fact that $p(x)$ is a two-valued function (with two branches usually denoted by $p_\pm(x)$). Here we choose to solve for $e^{-\sigma} = f_1(X_+)$ and $e^{\sigma} = f_2(X_-)$. We can write down two arbitrary expansions for $e^{\pm\sigma}$,

$$\begin{aligned} e^{-\sigma} = X_+ + g\left(\frac{\alpha}{2}X_+^{k+1} + \frac{\beta}{2}X_+^{-k+1}\right) + g^2\left(B_1 X_+ + B_2 X_+^{2k+1} + B_3 X_+^{-2k+1}\right) + \\ + g^3\left(B_4 X_+^{k+1} + B_5 X_+^{-k+1} + B_6 X_+^{3k+1} + B_7 X_+^{-3k+1}\right) \end{aligned} \tag{3.37}$$

and

$$\begin{aligned} e^{\sigma} = X_- + g\left(\frac{\alpha}{2}X_-^{-k+1} + \frac{\beta}{2}X_-^{k+1}\right) + g^2\left(\tilde{B}_1 X_- + \tilde{B}_2 X_-^{-2k+1} + \tilde{B}_3 X_-^{2k+1}\right) + \\ + g^3\left(\tilde{B}_4 X_-^{-k+1} + \tilde{B}_5 X_-^{k+1} + \tilde{B}_6 X_-^{-3k+1} + \tilde{B}_7 X_-^{3k+1}\right). \end{aligned} \tag{3.38}$$

The coefficients can be determined by plugging these ansatz into (3.35) and (3.36), with $a(\sigma)$ given by (3.34). Solving perturbatively in g, one finds

$$\begin{aligned} B1 &= \frac{\alpha\beta}{2} + \frac{d_3}{8}\,, \qquad B2 = \frac{\alpha^2(1+k)}{4} + \frac{d_1}{8}\,, \qquad B3 = \frac{\beta^2(1-k)}{4} + \frac{d_2}{8}\,, \\ B4 &= \alpha^2\beta\left(\frac{k^2+5k+6}{16}\right) + \alpha\left(\frac{2d_3 + kd_3}{16}\right) + \beta\left(\frac{2d_1 + kd_1}{16}\right) - d_4\,, \end{aligned}$$

$$B5 = \alpha\beta^2\left(\frac{k^2-5k+6}{16}\right)+\alpha\left(\frac{2d_2-kd_2}{16}\right)+\beta\left(\frac{2d_3-kd_3}{16}\right)-d_5\,,$$
$$B6 = \alpha^3\left(\frac{3k^2+5k+2}{16}\right)+\alpha\left(\frac{2d_1+3kd_1}{16}\right)-d_6\,,$$
$$B7 = \beta^3\left(\frac{3k^2-5k+2}{16}\right)+\beta\left(\frac{2d_2-3kd_2}{16}\right)-d_7\,, \tag{3.39}$$

and

$$\tilde{B}1 = \frac{\alpha\beta}{2}+\frac{d_3}{8}\,,\qquad \tilde{B}2=\frac{\alpha^2(1-k)}{4}+\frac{d_1}{8}\,,\qquad \tilde{B}3=\frac{\beta^2(1+k)}{4}+\frac{d_2}{8}\,,$$
$$\tilde{B}4 = \alpha^2\beta\left(\frac{k^2-5k+6}{16}\right)+\alpha\left(\frac{2d_3-kd_3}{16}\right)+\beta\left(\frac{2d_1-kd_1}{16}\right)d_4\,,$$
$$\tilde{B}5 = \alpha\beta^2\left(\frac{k^2+5k+6}{16}\right)+\alpha\left(\frac{2d_2+kd_2}{16}\right)+\beta\left(\frac{2d_3+kd_3}{16}\right)-d_5\,,$$
$$\tilde{B}6 = \alpha^3\left(\frac{3k^2-5k+2}{16}\right)+\alpha\left(\frac{2d_1-3kd_1}{16}\right)-d_6\,,$$
$$\tilde{B}7 = \beta^3\left(\frac{3k^2+5k+2}{16}\right)+\beta\left(\frac{2d_2+3kd_2}{16}\right)-d_7\,, \tag{3.40}$$

with

$$d_1 = \alpha^2\,(1+2k)\,,\qquad d_2=\frac{\beta^2}{\alpha^2}d_1\,,\qquad d_3=2\alpha\beta\,,\quad d_4=\alpha^2\beta\left(\frac{-8c_2-2k-3}{16}\right)$$
$$d_5 = \frac{\beta}{\alpha}d_4\,,\qquad d_6=\alpha^3\left(\frac{-8c_1-2k-1}{16}\right)\qquad d_7=\frac{\beta^3}{\alpha^3}d_6\,.$$

Notice that $\tilde{B}_i = B_i(k\to -k)$, as could have been seen directly from (3.37) and (3.38). Finally, to find the Fermi surface, we simply plug into

$$X_+X_- = a^2(e^{-\sigma},e^{\sigma}) \tag{3.41}$$

the explicit expressions $e^{-\sigma}=f_1(X_+)$ and $e^{\sigma}=f_2(X_-)$ given in (3.37) and (3.38). We find that the Fermi surface, to cubic order in perturbation theory, can be written as

$$\begin{aligned}X_+X_- &= 1-\frac{g}{2}\left[\alpha X_+^k+\alpha X_-^{-k}+\beta X_+^{-k}+\beta X_-^k\right]+\\ &+g^2\left[-\frac{3k}{4}\left(\alpha^2X_+^{2k}+\beta^2X_-^{2k}\right)+\frac{k}{4}\left(\alpha^2X_-^{-2k}+\beta^2X_+^{-2k}\right)\right]+\\ &+g^3\left[\left(-\frac{5k^2}{4}-\frac{k}{4}\right)\left(\alpha^3X_+^{3k}+\beta^3X_-^{3k}\right)+\left(-\frac{3k^2}{4}-\frac{k}{4}\right)\left(\alpha^2\beta X_+^k+\alpha\beta^2X_-^k\right)\right].\end{aligned}$$

It will turn out to be more convenient to rescale $k\to -k$ and to rewrite the surface in a slightly different way. After using $X_+X_- = 1-\frac{g}{2}[\alpha X_+^k+\alpha X_-^{-k}+\beta X_+^{-k}+\beta X_-^k]+\mathcal{O}(g^2)$ to manipulate the quadratic and cubic terms, we get

$$\begin{aligned}X_+X_- &= 1-\frac{g}{2}\left[\alpha X_+^{-k}+\alpha X_-^{k}+\beta X_+^{k}+\beta X_-^{-k}\right]+g^2\frac{k}{2}\left[\alpha^2X_+^{-2k}+\beta^2X_-^{-2k}\right]+\\ &+g^3\left[\left(-\frac{3k^2}{4}+\frac{k}{4}\right)\left(\alpha^3X_+^{-3k}+\beta^3X_-^{-3k}\right)+\left(-\frac{k^2}{4}+\frac{k}{4}\right)\left(\alpha^2\beta X_+^{-k}+\alpha\beta^2X_-^{-k}\right)\right].\end{aligned} \tag{3.42}$$

Moreover, recall that the time dependence was incorporated into the definition of $X_\pm$, $X_+ = e^{-t}(p+x)$ and $X_- = e^{t}(p-x)$. We can make the time dependence explicit by writing the surface in terms of x and p :

$$
\begin{aligned}
p^2 - x^2 = 1 &- \frac{g}{2}\left[\alpha e^{kt}\Big((p+x)^{-k} + (p-x)^{k}\Big) + \beta e^{-kt}\Big((p+x)^{k} + (p-x)^{-k}\Big)\right] + \\
&+ g^2 \frac{k}{2}\left[\alpha^2 e^{2kt}(p+x)^{-2k} + \beta^2 e^{-2kt}(p-x)^{-2k}\right] + \\
&+ g^3\left(-\frac{3k^2}{4} + \frac{k}{4}\right)\left[\alpha^3 e^{3kt}(p+x)^{-3k} + \beta^3 e^{-3kt}(p-x)^{-3k}\right] + \\
&+ g^3\left(-\frac{k^2}{4} + \frac{k}{4}\right)\left[\alpha^2\beta e^{kt}(p+x)^{-k} + \alpha\beta^2 e^{-kt}(p-x)^{-k}\right]. \qquad (3.43)
\end{aligned}
$$

4. Another approach to building backgrounds

Another approach for constructing non-trivial backgrounds and their corresponding Fermi surfaces was developed by AKK [13, 14]. It is based on the idea of associating a new, perturbed hamiltonian to each deformed Fermi surface. They found equations for the shape of the Fermi sea for a generic tachyon perturbation. It will be visible that this approach is very different from that of collective field theory. For the special case of the Sine-Liouville background, we will solve their equations, and find the explicit form of the Fermi surface. We will compare it to the Fermi surface that we obtained in the previous section, via collective field theory. It is non-trivial to show that the two approaches are equivalent.

We start by outlining the construction used in [13, 14]. Our calculations are then summarized in section 4.1. Before introducing the concept of a deformed Fermi surface, recall that the Fermi sea can be viewed as a collection of classical trajectories having energies less than some fixed, Fermi energy E_F. The ground state corresponds to the trajectory of the most energetic fermion, with energy E_F. For the standard inverted harmonic oscillator hamiltonian

$$H_0 = -\frac{1}{2}(\hat{X}_+\hat{X}_- + \hat{X}_-\hat{X}_+) \qquad (4.1)$$

the profile of the Fermi sea is

$$X_+X_- = \mu\,, \qquad (4.2)$$

where the Fermi energy was chosen to be $E_F = -\mu$.

Collective excitations are represented by deformations of the Fermi surface, which can be obtained by replacing μ by a general function of $X_\pm$,

$$X_+X_- = M(X_+, X_-)\,. \qquad (4.3)$$

While the ground state is stationary, the excited state corresponding to such a deformed surface will generically be time dependent. If we want to think of this perturbation as a new fermion ground state, we need to modify the one-fermion wave function to incorporate the non-trivial features of the deformed Fermi profile. Before discussing the perturbed wave function, it is useful to first discuss the case of the standard, inverted oscillator potential.

General solutions of the Schrodinger equation with hamiltonian (4.1), with a given energy E, take the form $\psi^E_\pm(X_\pm,t) = e^{-iEt}\,\psi^E_\pm(X_\pm)$, with

$$\psi^E_\pm(X_\pm) = \frac{1}{2\pi} X_\pm^{\pm iE-1/2}\,. \tag{4.4}$$

These functions obey standard completeness and orthonormality relations,

$$\begin{aligned} \langle \psi^E_\pm | \, \psi^{E'}_\pm \rangle \; &= \int_0^\infty dX_\pm \overline{\psi^E_\pm(X_\pm)}\, \psi^{E'}_\pm(X_\pm) = \delta(E-E')\,, \\ \int_{-\infty}^{\infty} dE \; \overline{\psi^E_\pm(X_\pm)}\, \psi^E_\pm(X'_\pm) \; &= \delta(X_\pm - X'_\pm)\,. \end{aligned} \tag{4.5}$$

Moreover, the X_+ and X_- representations can be related to each other by a unitary operator $\hat{S}$,

$$[\hat{S}\,\psi_+](X_-) = \int_0^\infty dX_+ K(X_-,X_+)\psi_+(X_+)\,, \tag{4.6}$$

where the Kernel K can be chosen to be

$$K(X_-,X_+) = \sqrt{\frac{2}{\pi}}\cos(X_- X_+)\,.$$

Notice that since the operator $\hat{S}$ relates incoming and outgoing waves, it can be interpreted as the fermionic scattering matrix. To see this more explicitly, note that

$$\left[\hat{S}^\pm\,\psi^E_\pm\right](X_\pm) = \frac{1}{\pi}\int_0^\infty dX_\pm \cos(X_+X_-)\, X_\pm^{\pm iE-1/2} = R(\pm E)\psi^E_\pm(X_\mp)\,, \tag{4.7}$$

with

$$R(E) = \sqrt{\frac{2}{\pi}}\cosh\left(\frac{\pi}{2}(\frac{i}{2}-E)\right)\,\Gamma(iE+1/2). \tag{4.8}$$

The factor $R(E)$, which is a pure phase, is the reflection coefficient for scattering off the inverted oscillator potential. Thus, scattering amplitudes between in and out states are given by

$$\langle \psi_- | \, \hat{S}\psi_+ \rangle = \langle \psi_- | \, K | \psi_+ \rangle \tag{4.9}$$

and in and out eigenfunctions satisfy the orthogonality relation

$$\langle \psi^E_- | \, K | \psi^{E'}_+ \rangle = R(E)\delta(E-E')\,. \tag{4.10}$$

So far we have restricted ourselves to the wave function of the standard inverted oscillator hamiltonian H_0. Next, we would like to generalize these arguments to less trivial backgrounds. If one wants to consider a perturbation such as $X_+X_- = M(X_+,X_-)$ as a new fermion *ground* state, the wave function needs to be modified. The perturbed wave functions $\Psi^E_\pm(X_\pm)$ can be related to the old, unperturbed ones by a phase factor,

$$\Psi^E_\pm(X_\pm) = e^{\pm i\varphi_\pm(X_\pm,E)}\psi^E_\pm(X_\pm)\,. \tag{4.11}$$

The authors of [13, 14] parametrize the phase in the following way:

$$\varphi_\pm(X_\pm;E) = \frac{1}{2}\phi(E) + V_\pm(X_\pm) + v_\pm(X_\pm;E)\,. \tag{4.12}$$

As will be shown later, the function $V_\pm(X_\pm)$ is what will be responsible for fixing the perturbation unambiguously. Notice that the phase $\phi(E)$ is precisely the analog of the reflection coefficient $R(E)$ discussed above, for the standard oscillator case. Here it has been explicitly incorporated in the expression for the new phase φ, and can be found by requiring

$$\hat{S}\Psi_+^E = \Psi_-^E \,. \tag{4.13}$$

Thus, the orthonormality of in and out eigenfunctions takes the form

$$\langle \Psi_-^{E_+} | \, K | \Psi_+^{E_-} \rangle = \delta(E_+ - E_-) \,, \tag{4.14}$$

and fixes the shape of the perturbed wave function.

This integral can be evaluated, at the quasiclassical level, by the saddle point approximation. Plugging in the wave functions, (4.14) takes the form

$$\begin{aligned} &\frac{1}{\pi\sqrt{2\pi}} \int dX_- X_-^{iE_- -1/2} e^{-i\varphi_-(X_-)} \int dX_+ X_+^{iE_+ -1/2} e^{-i\varphi_+(X_+)} e^{iX_+X_-} \\ &= \frac{1}{\pi\sqrt{2\pi}} \int dX_- X_-^{iE_- -1/2} e^{-i\varphi_-(X_-)} \int dX_+ X_+^{-1/2} e^{\ln X_+^{iE_+} - i\varphi_+(X_+) + iX_+X_-} . \end{aligned} \tag{4.15}$$

Minimizing the phase

$$\frac{\partial}{\partial X_+} \left(\ln X_+^{iE} - i\varphi_+(X_+) + iX_+X_- \right) = 0 \tag{4.16}$$

yields

$$X_+X_- = -E + X_+ \, \partial_+ \, \varphi_+(X_+) \,, \tag{4.17}$$

where we used the delta function $\delta(E_+ - E_-)$ and set $E \equiv E_+ = E_-$. Similarly, the saddle point approximation in the X_- integral gives

$$X_+X_- = -E + X_- \, \partial_- \, \varphi_-(X_-) \,. \tag{4.18}$$

Recalling that $E = -\mu$ and combining the two equations above, we get

$$X_+X_- = \mu + X_\pm \, \partial_\pm \, \varphi_\pm(X_\pm) \,, \tag{4.19}$$

which is precisely a deformed Fermi surface. Furthermore, notice that for the trivial case $\varphi_\pm = 0$ we recover the unperturbed Fermi surface,

$$X_+X_- = -E = \mu \,. \tag{4.20}$$

Thus, we have seen how the profile of the Fermi sea arises by imposing bi-orthogonality on the wavefunctions. Next, we would like to show how such surfaces are connected to deformed hamiltonians.

The functions $\Psi_\pm^E(X_\pm)$ are no longer eigenfunctions of the standard oscillator (4.1). If we introduce the perturbed hamiltonians $H_\pm$, associated with the $X_\pm$ representations, as solutions to the equation

$$H_\pm = H_0^\pm + X_\pm \partial_\pm \varphi_\pm(X_\pm, H_\pm) \,, \tag{4.21}$$

one can show that $\Psi^E_\pm(X_\pm)$ are eigenfunctions of $H_\pm$ with eigenvalue E. To see this, simply notice that

$$H_0^+\Psi_+(X_+) = -X_+\partial_{X_+}\varphi_+\Psi_+(X_+) + E\Psi_+(X_+)\,, \tag{4.22}$$

and similarly for $\Psi_-(X_-)$.

As one can see from the form of (4.19) and (4.21), in order to specify a given background one must fully specify the phase $\varphi_\pm$. In particular, it is enough to specify the form of the potentials $V_\pm$. For tachyon perturbations with momenta $p_n = n/R$, such potentials take the form

$$V_\pm(X_\pm) = R\sum_{k\geq 1} t_{\pm k} X_\pm^{k/R}\,. \tag{4.23}$$

Such a perturbation is exactly solvable, since it is generated by a system of commuting flows associated to the coupling constants $t_{\pm k}$. The resulting integrable structure is that of a constrained Toda lattice hierarchy. Here for the sake of simplicity we have chosen not to review this method, which is described in detail in [13, 14]. Instead, we have motivated the form of the Fermi surface by showing how it is connected to the perturbed hamiltonian and corresponding eigenfunctions.

Assuming that the phase $\varphi_\pm$ can be expanded as a Laurent series [13],

$$\varphi_\pm(X_\pm, E) = \frac{1}{2}\,\phi(E) + R\sum_{k\geq 1} t_{\pm k}\, X_\pm^{k/R} - R\sum_{k\geq 1}\frac{1}{k}\, v_{\pm k}\, X_\pm^{-k/R}\,, \tag{4.24}$$

and using $X_+X_- = \mu + X_\pm\,\partial_\pm\,\varphi_\pm$, we find

$$X_+X_- = \sum_{k\geq 1} k\, t_{\pm k}\, X_\pm^{k/R} + \mu + \sum_{k\geq 1} v_{\pm k}\, X_\pm^{-k/R}\,. \tag{4.25}$$

The unknown coefficients $v_{\pm k}$ and the phase ϕ are not independent, but are functions of $t_{\pm k}$ and μ (fixed by the bi-orthogonality condition); we will explicitly show how to find them for the Sine-Liouville case.

4.1 Fermi surface construction

The Sine-Liouville background is obtained by perturbing with the lowest couplings, $t_{\pm 1}$. The Fermi surface (4.19) can then be written in the two equivalent ways

$$X_+X_- \,=\, \mu + t_1 X_+^p + \sum_{k\geq 1} v_k X_+^{-pk}\,, \tag{4.26}$$

$$X_+X_- \,=\, \mu + t_{-1} X_-^p + \sum_{k\geq 1} v_{-k} X_-^{-pk}\,, \tag{4.27}$$

where we defined $p = 1/R$. Since we are interested in the comparison with the collective field theory Fermi surface, we need to solve for $v_{\pm k}$. To do so, we will use the convenient ansatz of AKK, given by

$$\begin{aligned} X_+ \,&=\, A\,\omega(1 + t_{-1} B\,\omega^{-p})\,, \\ X_- \,&=\, A\,\omega^{-1}(1 + t_1 B\,\omega^p)\,, \end{aligned} \tag{4.28}$$

with $A = e^{\frac{1}{2}\partial_\mu \phi(E)}$ and $B = A^{p-2}$. Plugging (4.28) into (4.26) and collecting equal powers of ω, one obtains equations for $v_{\pm k}$ as well as for the phase. Here we will stop at cubic order in perturbation theory, neglecting $\mathcal{O}(t_{\pm 1}^4)$ terms. Collecting the $\mathcal{O}(\omega^0)$ terms of (4.26) one finds the equation for the phase ϕ,

$$\mu e^{-\partial_\mu \phi} - (1-p) t_1 t_{-1} e^{-(2-p)\partial_\mu \phi} = 1\,, \tag{4.29}$$

which appears in [13, 14].

Expanding $A = e^{\frac{1}{2}\partial_\mu \phi(E)}$ perturbatively, we find

$$A = e^{\frac{1}{2}\partial_\mu \phi(E)} = \sqrt{\mu} + t_1 t_{-1} \frac{p-1}{2} \mu^{p-3/2} + \mathcal{O}(t^4)\,.$$

Collecting the ω^{-p} powers of (4.26) gives

$$v_1 = \mu^p\, t_{-1} + t_1\, t_{-1}^2\, \mu^{2p-2}\, \frac{p\,(p-1)}{2} + \mathcal{O}(t^4)\,,$$

while v_2 is obtained by looking at the ω^{-2p} terms,

$$v_2 = p\, t_{-1}^2\, \mu^{2p-1} + \mathcal{O}(t^4)\,.$$

Finally, the ω^{-3p} terms yield

$$v_3 = t_{-1}^3\, \mu^{3p-2}\, \frac{p\,(3p-1)}{2}. \tag{4.30}$$

Similarly, the coefficients v_{-k} are found by plugging the ansatz (4.28) into (4.27). This yields

$$\begin{aligned} v_{-1} &= \mu^p\, t_1 + t_1^2\, t_{-1}\, \mu^{2p-2}\, \frac{p\,(p-1)}{2}\,, \\ v_{-2} &= p\, t_1^2\, \mu^{2p-1}\,, \\ v_{-3} &= t_1^3\, \mu^{3p-2}\, \frac{p\,(3p-1)}{2}\,, \end{aligned}$$

from the ω^p,ω^{2p} and ω^{3p} terms respectively.

Having found the coefficients $v_{\pm k}$, we can add the two Fermi surfaces (4.26) and (4.27) to get the following:

$$\begin{aligned} X_+ X_- &= \mu + \frac{1}{2}\left(t_1 X_+^p + t_{-1} X_-^p + \mu^p t_1 X_-^{-p} + \mu^p t_{-1} X_+^{-p}\right) + \\ &+ \frac{p}{2}\left(t_1^2\, \mu^{2p-1}\, X_-^{-2p} + t_{-1}^2\, \mu^{2p-1}\, X_+^{-2p}\right) + \\ &+ \frac{p\,(p-1)}{4}\, \mu^{2p-2}\left(t_1\, t_{-1}^2 X_+^{-p} + t_1^2\, t_{-1} X_-^{-p}\right) + \\ &+ \frac{p\,(3p-1)}{4}\, \mu^{3p-2}\, \left(t_1^3 X_-^{-3p} + t_{-1}^3 X_+^{-3p}\right) + \mathcal{O}(t^4)\,. \end{aligned} \tag{4.31}$$

5. Comparison

To make the comparison with the collective field theory surface (3.42) more explicit, we can rewrite (4.31) in a more suggestive form,

$$\begin{aligned}
X_+X_- &= \mu + \frac{1}{2}\sqrt{\mu}^p \left[t_1 \left(\frac{X_+}{\sqrt{\mu}}\right)^p + t_{-1}\left(\frac{X_-}{\sqrt{\mu}}\right)^p + t_1\left(\frac{X_-}{\sqrt{\mu}}\right)^{-p} + t_{-1}\left(\frac{X_+}{\sqrt{\mu}}\right)^{-p}\right] + \\
&\quad + \frac{p}{2}\sqrt{\mu}^{2p-2}\left[t_1^2\left(\frac{X_-}{\sqrt{\mu}}\right)^{-2p} + t_{-1}^2\left(\frac{X_+}{\sqrt{\mu}}\right)^{-2p}\right] + \\
&\quad + \sqrt{\mu}^{3p-4}\,\frac{p\,(p-1)}{4}\left[t_1\, t_{-1}^2\left(\frac{X_+}{\sqrt{\mu}}\right)^{-p} + t_1^2\, t_{-1}\left(\frac{X_-}{\sqrt{\mu}}\right)^{-p}\right] + \\
&\quad + \sqrt{\mu}^{3p-4}\frac{p\,(3p-1)}{4}\left[t_1^3\left(\frac{X_-}{\sqrt{\mu}}\right)^{-3p} + t_{-1}^3\left(\frac{X_+}{\sqrt{\mu}}\right)^{-3p}\right] + \mathcal{O}(t^4)\,. \qquad (5.1)
\end{aligned}$$

Defining new, rescaled coordinates

$$x_+ = \frac{X_+}{\sqrt{\mu}}\,, \qquad x_- = \frac{X_-}{\sqrt{\mu}}\,,$$

the above Fermi surface becomes

$$\begin{aligned}
x_+x_- &= 1 + \frac{1}{2}\sqrt{\mu}^{p-2}\left(t_1x_+^p + t_{-1}x_-^p + t_1x_-^{-p} + t_{-1}x_+^{-p}\right) + \\
&\quad + \frac{p}{2}\sqrt{\mu}^{2p-4}\left(t_1^2\,x_-^{-2p} + t_{-1}^2\,x_+^{-2p}\right) + \\
&\quad + \sqrt{\mu}^{3p-6}\left[\frac{p\,(p-1)}{4}\left(t_1\,t_{-1}^2x_+^{-p} + t_1^2\,t_{-1}x_-^{-p}\right) + \frac{p\,(3p-1)}{4}\left(t_1^3x_-^{-3p} + t_{-1}^3x_+^{-3p}\right)\right].
\end{aligned}$$

Finally, letting $t_{-1} = -\alpha, t_1 = -\beta$, it takes the form

$$\begin{aligned}
x_+x_- &= 1 - \frac{1}{2}\left(\sqrt{\mu}^{p-2}\right)\left(\alpha x_-^p + \alpha x_+^{-p} + \beta x_+^p + \beta x_-^{-p}\right) + \\
&\quad + \frac{p}{2}\left(\sqrt{\mu}^{p-2}\right)^2\left(\alpha^2\,x_+^{-2p} + \beta^2\,x_-^{-2p}\right) + \left(\sqrt{\mu}^{p-2}\right)^3 \times \\
&\quad \times \left[\frac{p\,(-p+1)}{4}\left(\beta\,\alpha^2x_+^{-p} + \beta^2\,\alpha x_-^{-p}\right) + \frac{p\,(-3p+1)}{4}\left(\alpha^3x_+^{-3p} + \beta^3x_-^{-3p}\right)\right],
\end{aligned}$$

which exactly matches the collective field theory surface (3.42) provided one identifies g with $\sqrt{\mu}^{p-2}$ and k with p. Thus, we have seen that the collective field construction, which starts from the standard inverted oscillator potential, yields the same Fermi sea profile (to cubic order in perturbation theory) as the method of AKK, which on the other hand made use of perturbed hamiltonians.

6. Conclusions

A lot of recent research has been focusing on the construction of time-dependent backgrounds in 2D string theory. Some of these non-trivial backgrounds have been obtained by replacing the standard Liouville interaction by a more general momentum or winding perturbation. In particular, the Sine-Liouville background, which is a simple example of a momentum perturbation, has received significant attention. The Sine-Liouville model has been studied in two apparently different ways; one is based on the collective field theory description of 2D string theory, while the other uses the tools of Toda integrable systems. While in the collective field approach the hamiltonian is that of the standard inverted harmonic oscillator, the second method models the tachyon perturbation by introducing certain deformed hamiltonians. In this work we have analyzed the two approaches to obtaining the Sine-Liouville background, and we have constructed in each case the explicit form of the Fermi surface. Comparison of the two resulting surfaces demonstrates agreement between the seemingly distinct methods, and sheds some light on the connection between them. We would like to note that the collective field method allows for the construction of non-trivial backgrounds without the need to introduce deformed hamiltonians, at least for the types of perturbations considered here. The issue of which backgrounds can be generated by the introduction of appropriate non-trivial hamiltonians is still an interesting one, which we would like to further investigate.

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